MUNI
FACULTY
OF SCIENCE

Reviewer's Dissertation Thesis Report

Masaryk University Faculty of Science	
Doctoral Study Field	Mathematics and Statistics, specialization Probability, Statistics and Mathematical Modelling
Student	***Mgr. Jakub Záthurecký***
Disertation Thesis Title	*Dynamical models in neuroscience*
Reviewer	dr. Babette de Wolff
Reviewer's Workplace	Vrije Universiteit Amsterdam

Text of the report (its extent is on reviewer's consideration) ...
The thesis submitted by Jakub Záthurecký is concerned with bifurcations of periodic orbits, with an emphasis on period doubling bifurcations. Sufficient criteria for the occurrence of a period doubling bifurcation and criteria to determine the stability exchange are provided for ordinary differential equations (Chapter 1), retarded functional differential equations (Chapter 2) and neutral functional differential equations (Chapter 4). Chapter 3 provides a case study of pattern formation in a network of delay-coupled neurons.

The overarching theme of the thesis is that the bifurcation statements are proven by means of Lyapunov-Schmidt reduction on spaces of periodic functions. This approach to proving bifurcation statements had (in different settings) previously been used by Golubitsky and Stewart, but it differs from the most predominantly used approach, which is based on center manifold reduction. Using the Lyapunov-Schmidt approach, Jakub Záthurecký gives an alternative proof to an already existing result (period doubling bifurcations in ordinary differential equations) and additionally proves two novel results (period doubling bifurcations in retarded functional differential equations and in neutral functional differential equations with stable D-operator).

What struck me most about the thesis, is that it makes a very convincing case for the Lyapunov-Schmidt reduction approach to bifurcation theory. The center manifold approach, which is currently more standard, becomes relatively involved in the context of retarded functional differential equations, where one needs additional functional analytical machinery to be able to perform this center manifold reduction. In contrast, Jakub Záthurecký's work shows how flexible the Lyapunov-Schmidt reduction

approach is, and how relatively seamlessly (most) arguments can be lifted from ordinary differential equations to retarded and even neutral functional differential equations.

My points of criticism have to do with the exposition of the results. The chapters could have concluded with clearer statements of the main results – formulating the statements so that they could be independently read from the rest of the text would have been beneficial. Also in the formulation of the lemma's could have been sharper (at points, the most important message of the lemma was actually formulated after the lemma). More content-wise, I felt that Chapter 3 – although interesting in itself – was somewhat disconnected from the rest of the thesis, since it used Lyapunov-Schmidt reduction but is seemed not to be concerned with period doubling bifurcations.

In my opinion, the thesis clearly fulfills the necessary requirements to grant Jakub Záthurecký the doctoral degree. The thesis shows that Jakub Záthurecký has strong technical skills which he uses to prove valuable contributions to applied mathematics. Taking a step back from results to more general attitude, it is praiseworthy that Jakub Záthurecký evidently took care choosing a suitable approach to proving them.

Potential questions for the defense (in no particular order):

General questions:

1. The candidate takes an approach to proving bifurcation theorems which uses Lyapunov-Schmidt reduction on spaces of periodic functions, rather than performing a center manifold reduction in the state space. Can the candidate elaborate on how these two approaches compare, i.e. what are the advantages and limitations of the respective approaches? Does the candidate expect that every result that is obtained by center manifold reduction also be obtained by the Lyapunov-Schmidt approach and vice versa?
2. In Chapter 3, the candidate writes that similar results might be obtained by applying the Equivariant Branching Lemma, but that its application is not straightforward. Can the candidate elaborate on what the hurdles are in applying the Equivariant Branching Lemma in this case?

More technical questions:
3. In Chapter 3, the candidate assumes that K is an equivariant operator (p. 58). Can the candidate give an explicit example of a coupling function that satisfies this equivariance condition (say for three oscillators)? Does it imply that the coupling delays between all the nodes is the same?
4. The discussion on page 36 relies on the use of the formal adjoint equation (2.6). Suppose one would not know that the formal adjoint equation exists, but would just define the functions v_1 and v_2 as the left eigenvectors of the monodromy operator T(0, p), could one then still prove Theorem 22?

Conclusion

In the dissertation thesis, the student has demonstrated his/her creative abilities in the relevant research area:

YES ☒ NO ☐

I am recommending the dissertation thesis for defence:

YES ☒ NO ☐

Date 7 February 2024

.....
Signature

Masaryk University, Faculty of Science

Kotlářská 2, 611 37 Brno, Czech Republic
E: veda@sci.muni.cz, www.sci.muni.cz
Bank account: KB Brno, Ref. No.: 85636621/0100, ID: 00216224, Tax ID: CZ00216224